EINSTEIN METRICS AND THE NUMBER OF SMOOTH STRUCTURES ON A FOUR–MANIFOLD

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ABSTRACT. We prove that for every natural number k there are simply connected topological four–manifolds which have at least k distinct smooth structures supporting Einstein metrics, and also have infinitely many distinct smooth structures not supporting Einstein metrics. Moreover, all these smooth structures become diffeomorphic to each other after connected sum with only one copy of the complex projective plane. We prove that manifolds with these properties cover a large geographical area.

1. INTRODUCTION

All the classical obstructions to the existence of Einstein metrics on four–manifolds are homotopy invariant. If a closed orientable 4–manifold M admits an Einstein metric, then its Euler characteristic has to be non–negative, and, furthermore, the Hitchin–Thorpe inequality

$$e(M) \geqslant \frac{3}{2}|\sigma(M)| \tag{1}$$

must hold [15], where e denotes the Euler characteristic and σ the signature. This condition is clearly homotopy invariant, as are the restrictions coming from Gromov's notion of simplicial volume [14, 20], and from the existence of maps of non–zero degree to hyperbolic manifolds [38].

Using Seiberg–Witten invariants, LeBrun [24] gave the first examples of simply connected smooth four-manifolds which satisfy the (strict) Hitchin–Thorpe inequality, but still do not admit Einstein metrics. As his examples were not known to be homeomorphic to manifolds admitting Einstein metrics, LeBrun's paper implicitly raised the question whether the new obstruction might in fact be homotopy invariant, or not. This issue was disposed

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of by the second author in [19]. Using LeBrun's [24] work, Kotschick [19] proved the following result, showing for the first time that the smooth structures of 4–manifolds form definite obstructions to the existence of an Einstein metric.

Theorem 1. *There are infinitely many pairs (X_i, Z_i) of simply connected closed oriented smooth 4–manifolds such that:*

1) *X_i is homeomorphic to Z_i,*
2) *if $i \neq j$, then X_i and X_j are not homotopy equivalent,*
3) *Z_i admits an Einstein metric but X_i does not,*
4) $e(X_i) > \frac{3}{2}|\sigma(X_i)|$.

Note that 3) implies in particular that X_i and Z_i are not diffeomorphic.

After the proof of Theorem 1, Kotschick asked how many smooth structures with Einstein metrics and how many without such metrics exist on a given topological manifold, see [19] p. 6-7. He pointed out that, using for example the work of Fintushel–Stern, one can show that one has infinitely many choices for the smooth structures of the manifolds X_i in Theorem 1. Kotschick [19] also remarked that by the work of Salvetti [37], the number of distinct smooth structures among sets of homeomorphic minimal surfaces of general type can be arbitrarily large, and that all the examples in [37] have ample canonical bundle, and therefore have Kähler–Einstein metrics of negative scalar curvature. Thus, the number of smooth structures admitting Einstein metrics can be arbitrarily large.

The purpose of this paper is to show that these two phenomena, infinitely many smooth structures without Einstein metrics and an arbitrarily large number of smooth structures with Einstein metrics, can be realized on the same topological manifold. We shall prove the following:

Theorem 2. *For every natural number k there is a simply connected topological 4-manifold M_k which has at least k distinct smooth structures Z_k^i supporting Einstein metrics, and also has infinitely many distinct smooth structures X_k^j not supporting Einstein metrics.*

Moreover, all the $Z_k^i \# \mathbb{C}P^2$ and $X_k^j \# \mathbb{C}P^2$ for fixed k are diffeomorphic to each other.

We shall produce lots of such examples, with ratios $|\sigma|/e$ which are dense in the interval $[\frac{1}{3}, \frac{1}{2}]$, compare Theorem 7 in Section 4.

As the Z_k^i and X_k^j are all homeomorphic for fixed k, Wall's classical result [42] implies that they are stably diffeomorphic. That a single stabilization with $\mathbb{C}P^2$ suffices can be interpreted to mean that these differentiable structures are as close to each other as is possible while still being non-diffeomorphic. In fact, we shall exhibit Z_k^i and X_k^j as in Theorem 2

which are almost completely decomposable (ACD) in the sense of Mandelbaum [29], so that their connected sums with $\mathbb{C}P^2$ are diffeomorphic to $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ for some p and q. Whether such a decomposable manifold can admit an Einstein metric is only known in very few cases with $p = 1$.

To put Theorem 2 into perspective, we continue with the chronology of earlier work in this direction. After [19] appeared, LeBrun [25, 26] refined his arguments from [24], and produced more examples of precisely the type exhibited in Theorem 1, where one has pairs of homeomorphic manifolds such that one is Einstein and the other is not. However, he did not discuss the number of smooth structures. This was taken up recently by Ishida and LeBrun in [16]. They give examples of simply connected topological four-manifolds with an infinite number of smooth structures which do not admit Einstein metrics. Like LeBrun in his earlier papers [24, 25, 26], Ishida–LeBrun [16] do not exhibit multiple smooth structures with Einstein metrics on the same manifold where one has infinitely many smooth structures without such metrics. In fact, for the most interesting ones of their examples, no smooth structure with an Einstein metric is known.

One of the difficulties in proving results like Theorems 1 and 2 above is that there are almost no existence results for Einstein metrics on simply connected 4-manifolds. Therefore, one is always forced to arrange a situation where one can appeal to the only existence result covering lots of homeomorphism types, which is the resolution of the Calabi conjecture for negative scalar curvature due to Aubin [1] and Yau [44]. This then leads to questions about the geography of complex surfaces of general type, and of some related classes of four-manifolds. Thus, in the present paper we make substantial progress on two geographical questions, which are of interest independently of the applications to Einstein metrics. One is the geography of algebraic surfaces which are iterated branched covers of the plane, the other is about symplectic four-manifolds which are almost completely decomposable.

Salvetti [37] considered iterated cyclic branched covers of the projective plane, and used these to prove that for any k, there exists a pair of invariants e and σ such that for this pair one has at least k homeomorphic surfaces with different divisibilities for their canonical classes. In his examples, the ratios σ/e are so close to zero that one cannot use them to prove Theorem 2 with the arguments of [24, 19]. In fact, even the improved estimates of [25, 26] do not apply. Therefore, in Section 2 below we provide a generalization of Salvetti's arguments which shows that, by choosing the parameters judiciously, iterated cyclic branched covers of the projective plane can be used to cover other parts of the geography of surfaces. In particular,

we can arrange k–tuples of homeomorphic surfaces with different divisibilities for the canonical class with characteristic numbers which are such that homeomorphic manifolds without Einstein metrics can be found using the improved estimate from [26].

In Section 3, which is inspired in part by the work of J. Park [35], we discuss the geography of minimal symplectic four-manifolds which are almost completely decomposable. Blowups of these will be used for the X_k^j in Theorem 2. Even without the ACD requirement, our geography results are qualitatively stronger than what was known before, compare for example [12, 35, 36].

In Section 4 we combine the different ingredients to prove a more precise version of Theorem 2. We shall also exhibit infinitely many smooth structures without Einstein metrics on many other manifolds which are not homotopy equivalent to complex surfaces, for which the existence of smooth structures with Einstein metrics is an open question. See Theorem 8.

In Section 5 we give explicit examples of manifolds with very small homology which have a smooth structure supporting an Einstein metric and have infinitely many smooth structures which do not support such a metric. We also give simple explicit examples with multiple smooth structures admitting Einstein metrics.

2. The geography of iterated branched covers of $\mathbb{C}P^2$

In this section we study the spread of Chern numbers among algebraic surfaces which are iterated branched covers of the projective plane. We build on the work of Salvetti [37] to show that for any integer k there are k-tuples of homeomorphic surfaces with ample canonical classes of different divisibilities, whose ratio c_1^2/χ can be specified arbitrarily within a certain range. Note that k-tuples of homeomorphic surfaces with canonical classes of different divisibilities were first exhibited by Catanese [5] using bidouble covers of $\mathbb{C}P^1 \times \mathbb{C}P^1$. In his examples the divisibilities are even, but the method probably extends to odd divisibility. Nevertheless, we found iterated covers of the plane to be more convenient to use.

Given positive integers r, $d_1, \ldots, d_r$ and $m_1, \ldots, m_r$, one can construct a simply-connected complex algebraic surface S by starting from the projective plane and repeatedly passing to coverings of degrees d_j branched along the preimages of smooth curves of degree $n_j = d_j m_j$ in the plane. The canonical class of S is $\sum_{j=1}^{r}(d_j - 1)m_j - 3$ times the pullback of the class of a line in the projective plane. Except for some small values of the parameters, the surface S so obtained is minimal of general type and has

ample canonical bundle. The Chern numbers of S are

$$\begin{aligned} c_1^2(S) &= d_1 \cdot \ldots \cdot d_r (\sum_{j=1}^{r} (d_j - 1) m_j - 3)^2 \, , \\ c_2(S) &= \frac{1}{2} d_1 \cdot \ldots \cdot d_r \big((\sum_{j=1}^{r} (d_j - 1) m_j - 3)^2 + (\sum_{j=1}^{r} (d_j^2 - 1) m_j^2 - 3) \big) \, , \end{aligned}$$

and they determine the holomorphic Euler characteristic $\chi(S)$ and the signature

$$\sigma(S) = -\frac{1}{3} d_1 \cdot \ldots \cdot d_r (\sum_{j=1}^{r} (d_j^2 - 1) m_j^2 - 3) \, .$$

We consider the inverse problem, starting with a fixed pair of invariants, say $c_1^2(S)$ and $\sigma(S)$, and try to find k solutions of the above equations for $d_1, \ldots, d_r$ and $m_1, \ldots, m_r$. Salvetti [37] considered the special case when the covering degrees d_j are all equal and the m_j are not too far from being equal; more precisely he assumed $\sum_{j=1}^{r} m_j^2 \leqslant \frac{1}{r-1} (\sum_{j=1}^{r} m_j)^2$. This leads to ratios for $c_1^2(S)/\chi(S)$ close to 8, which is not suitable for our purposes. In order to use these surfaces as the Z_k^i in Theorem 2, we need $c_1^2(S)/\chi(S)$ to be somewhere below 6, and the smaller we get this ratio, the easier the proof will be. In order to minimize $c_1^2(S)/\chi(S)$ we have to maximize the quotient of $\sum (d_j^2 - 1) m_j^2$ by $(\sum (d_j - 1) m_j)^2$, i. e. we should have a few of the m_j much bigger than the others and the corresponding covering degrees d_j small.

We can do better and adjust $c_1^2(S)/\chi(S)$ to approximate any value between 4 and 8.

Theorem 3. *Let k be a positive integer. There are values for c_1^2 and σ which are realized by at least k iterated branched covers with different divisibilities of the canonical class. The divisibilities can be arranged to be all even, or all odd. The corresponding ratios c_1^2/χ are dense in the interval* $[4, 8]$.

Proof. To spread out the Chern numbers, fix rational numbers $\mu_1, \ldots, \mu_s$ normalized by $\mu_1 + \ldots + \mu_s = 1$. Put $\mu^2 = \sum_{j=1}^{s} \mu_j^2$ and note that by a suitable choice of s and μ_j we can place μ^2 anywhere between 0 and 1 because the set of numbers of the form $\sum_{j=1}^{s} \mu_j^2$ with arbitrary s and rational $\mu_1, \ldots, \mu_s$ summing up to 1 is dense in the unit interval.

Consider a tower of coverings of the projective plane by s iterated double covers branched over curves of degree $2\mu_j m_0$, where the integer m_0 will be fixed later. Of course m_0 has to be a multiple of the denominators of $\mu_1, \ldots, \mu_s$. In the end we will let m_0 grow to infinity.

Now on top of this tower we consider 16 further cyclic covers of very high degree d branched over the preimages of curves of degrees $dm_1, \ldots, dm_{16}$.

If m_0 is sufficiently large, the elementary number theory worked out by Salvetti [37] will provide us with several solutions for $(d, m_1, \ldots, m_{16})$ giving rise to the same invariants c_1^2 and σ for the total covering surfaces.

For our $s + 16$ stage tower, the formulae for the invariants specialize to

$$c_1^2(S) = 2^s d^{16}(m_0 + (d-1)\sum_{j=1}^{16} m_j - 3)^2 \tag{2}$$

$$\sigma(S) = -\frac{1}{3} 2^s d^{16}(3\mu^2 m_0^2 + (d^2-1)\sum_{j=1}^{16} m_j^2 - 3) \ . \tag{3}$$

Fix $\delta > 0$. Since the sum $\sum_p p^{-1}$ over prime numbers p diverges, for any real number $\alpha > 1$ the number of primes between α^n and α^{n+1} will be unbounded for $n \to \infty$. Hence we can find $k+1$ odd primes such that the largest one is at most α times the smallest one. Forgetting the smallest one, we obtain a set D of k odd primes such that $(d+1)/(d'-1) < \alpha$ for any $d, d' \in D$. Note that D depends on α, though this is not explicit in our notation. Put $d^* = \min(D)$ and $\varepsilon = \delta/(d^*-1)$. It is clear that ε converges to 0 for $\alpha \to 1$ and δ fixed.

These $d \in D$ will be used as degrees in our tower of coverings. Put $P = \prod_{d \in D} d$. By (2) and since the $d \in D$ are primes, we can write $c_1^2 = 2^s P^{16} Q$ for some integer Q. Furthermore, since $c_1^2/2^s d^{16}$ has to be a square, we can write $Q = C^2$ for an integer C. Similarly, we can write $\sigma = -\frac{1}{3} 2^s P^{16} C'$ for some integer C'. The equations (2) and (3) now read

$$(d-1)\sum_{j=1}^{16} m_j = (P/d)^8 C - (m_0 - 3) \ , \tag{4}$$

$$(d^2-1)\sum_{j=1}^{16} m_j^2 = (P/d)^{16} C' - 3(\mu^2 m_0^2 - 1) \ . \tag{5}$$

We are left with the task of solving the pair of equations

$$\sum_{j=1}^{16} m_j = A_d, \qquad \sum_{j=1}^{16} m_j^2 = B_d \tag{6}$$

for each $d \in D$ (separately), where we have put

$$A_d = ((P/d)^8 C - (m_0 - 3))/(d-1) \ ,$$
$$B_d = ((P/d)^{16} C' - 3(\mu^2 m_0^2 - 1))/(d^2 - 1) \ .$$

We can achieve that both A_d and B_d are integers by the following

Lemma 1. *There are integers C, C' such that for every $d \in D$ we have $(P/d)^8 C \equiv m_0 - 3 \bmod d-1$ and $(P/d)^{16} C' \equiv 3(\mu^2 m_0^2 - 1) \bmod d^2 - 1$.*

Proof. Since D consists of nearby primes we infer that d and $d' \pm 1$ are coprime for any $d, d' \in D$. Hence P is invertible $\mod \prod_d (d-1)$ and we can find C with $P^8 C \equiv m_0 - 3 \bmod \prod_d (d-1)$. Since $(P/d)^8 \equiv P^8 \bmod d-1$ for every d the result follows.

Similarly P is invertible mod $\prod_d (d^2-1)$ so we find C' with $P^{16}C' \equiv 3(\mu^2 m_0^2 - 1) \bmod \prod_d (d^2-1)$. Since $d^{16} \equiv 1 \bmod d^2-1$ the result follows. □

We are free to modify C by a multiple of $\prod_d (d-1)$ and C' by a multiple of $\prod_d (d^2-1)$; hence we can arrange that A_{d^*} and B_{d^*} have distance less than P^9 respectively P^{18} from any given values A and B. Notice that with C and C' satisfying Lemma 1 both A_d and B_d will be even for every d (as soon as $k \geqslant 2$).

Our goal is to choose C, C' in such a way that we can solve the pair of equations (6) relying on the following

Lemma 2 (Salvetti [37]). *Let A, r be integers with $A > 0$ and $r \geqslant 16$. The integral quadratic form $x_1^2 + \ldots + x_r^2$, under the restriction $x_1 + \ldots + x_r = A$ and $x_j > 0$ for $j = 1, \ldots, r$, represents all integers $B \equiv A \pmod 2$ such that*

$$A^2/r + \alpha_r \leqslant B \leqslant A^2/(r-1),$$

where α_r is a constant depending only on r.

Proof. This is the Lemma on page 166 of [37]. The proof is elementary, using that each non-negative integer is a sum of four squares. □

Recall that we denote the smallest element of D by d^*. For any positive integer m_0 choose $C(m_0)$ such that

$$A_{d^*} = ((P/d^*)^8 C(m_0) - (m_0 - 3))/(d^* - 1)$$

satisfies $|A_{d^*} - \varepsilon m_0| < P^9$. Since P does not depend on m_0, we will eventually have

$$\tfrac{1}{2}\varepsilon^2 m_0^2 + 240\alpha_{16} < A_{d^*}^2 < (2\varepsilon m_0)^2$$

for all sufficiently large m_0, where α_{16} is the constant from Lemma 2 with $r = 16$. Then the quantity

$$\Delta(m_0) = \frac{A_{d^*}^2}{15} - \left(\frac{A_{d^*}^2}{16} + \alpha_{16}\right) = \frac{A_{d^*}^2}{240} - \alpha_{16}$$

will be bounded below by

$$\Delta(m_0) > \frac{\varepsilon^2}{480} m_0^2 \tag{7}$$

for large m_0. Now choose $C'(m_0)$ such that B_{d^*} differs from $A_{d^*}^2/16 + \Delta(m_0)/2$ by no more than P^{18}. Then for large m_0 we will have

$$\frac{A_{d^*}^2}{16} + \alpha_{16} + \frac{1}{3}\Delta(m_0) < B_{d^*} < \frac{A_{d^*}^2}{15} - \frac{1}{3}\Delta(m_0)\ . \tag{8}$$

Lemma 3. *If α is sufficiently close to 1 then for all $m_0 \gg 0$ we have*

$$\frac{A_d^2}{16} + \alpha_{16} < B_d < \frac{A_d^2}{15}$$

for every $d \in D$.

Proof. We first show that A_d^2 differs from $A_{d^*}^2$ by no more than $\Delta(m_0)/6$. To see this, observe that

$$\begin{aligned} |(d-1)A_d - (d^*-1)A_{d^*}| &= (P/d^*)^8 C(m_0)|(d^*/d)^8 - 1| \\ &\leqslant (\alpha^8 - 1)((d^*-1)A_{d^*} + m_0 - 3)\ . \end{aligned}$$

We divide by $d-1$ and use $|(d^*-1)/(d-1) - 1| < \alpha - 1$ to obtain

$$|A_d - A_{d^*}| \leqslant (\alpha^8 - 1)(A_{d^*} + \frac{m_0 - 3}{d-1}) + (\alpha - 1)A_{d^*}$$

or, using $A_{d^*} < 2\varepsilon m_0$ and $1/(d-1) \leqslant \varepsilon/\delta$,

$$|A_d - A_{d^*}| \leqslant h(\alpha)\varepsilon m_0$$

with $h(\alpha) = (\alpha^8 - 1)(2 + 1/\delta) + 2(\alpha - 1)$, which goes to zero when $\alpha \to 1$. Then

$$|A_d^2 - A_{d^*}^2| \leqslant 2A_{d^*}|A_d - A_{d^*}| + |A_d - A_{d^*}|^2 \leqslant h'(\alpha)\varepsilon^2 m_0^2$$

where $h'(\alpha) = 4h(\alpha) + h(\alpha)^2$ also goes to zero when $\alpha \to 1$. This shows that as soon as α is closer to 1 than some constant depending only on δ, the difference $|A_d^2 - A_{d^*}^2|$ is less than 1/6 times the term on the right-hand side of (7).

A similar computation shows that we can assume the same bound for $|B_d - B_{d^*}|$. In detail,

$$\begin{aligned} |(d^2-1)B_d - (d^{*2}-1)B_{d^*}| &\leqslant (P/d^*)^{16} C'(m_0)|(d^*/d)^{16} - 1| \\ &\leqslant (\alpha^{16} - 1)((d^{*2}-1)B_{d^*} + 3\mu^2 m_0^2 - 3)\ . \end{aligned}$$

Dividing by $d^2 - 1$ we obtain

$$|B_d - B_{d^*}| \leqslant (\alpha^{16} - 1)(B_{d^*} + \frac{3\mu^2 m_0^2 - 3}{d^2 - 1}) + (\alpha^2 - 1)B_{d^*}\ .$$

Using $B_{d^*} < \frac{1}{15}A_{d^*}^2 < \frac{4}{15}\varepsilon^2 m_0^2$ and $1/(d^2-1) \leqslant (\varepsilon/\delta)^2$ this gives $|B_d - B_{d^*}| < h''(\alpha)\varepsilon^2 m_0^2$ where $h''(\alpha) = (\alpha^8 - 1)(\frac{4}{15} + 3\mu^2/\delta^2) + \frac{4}{15}(\alpha^2 - 1)$ becomes arbitrarily small as $\alpha \to 1$.

Now the claim follows from the estimates (8) because replacing d^* by d does not change any of the terms by more than $\Delta(m_0)/6$. □

According to Lemma 2 this shows that for each $d \in D$ the pair of equations (6) is solvable by positive integers.

Next we calculate the ratios c_1^2/χ for these surfaces. As they all have the same invariants, we can look at the one with the reference parameter $d = d^*$. For the degrees of the branch divisors we have the estimate $\sum_{j=1}^r m_j = A_d \leqslant 2\varepsilon m_0$. This gives

$$\begin{aligned} c_1^2 &= 2^s d^{16}(m_0 + (d-1)\sum_{j=1}^{16} m_j - 3)^2 \\ &= 2^s d^{16} m_0^2 (1 + (d-1)O(\varepsilon))^2 \ = \ 2^s d^{16} m_0^2 (1 + O(\delta))^2 \ , \end{aligned}$$

and, using $\sum_j m_j^2 \leqslant (\sum_j m_j)^2$,

$$\begin{aligned} -\sigma \ &= \ \tfrac{1}{3} 2^s d^{16}(3\mu^2 m_0^2 + (d^2-1)\sum_{j=1}^{16} m_j^2 - 3) \\ &= \ 2^s d^{16} \mu^2 m_0^2 (1 + (d^2-1)O(\varepsilon^2)) \ = \ 2^s d^{16}\mu^2 m_0^2 (1 + O(\delta^2)) \ . \end{aligned}$$

Since we can choose δ arbitrarily small, then α arbitrarily close to 1 (increasing d) and, finally, m_0 arbitrarily large, these estimates for c_1^2 and σ show that we can arrange $-\sigma/c_1^2$ arbitrarily close to μ^2.

Recall that the possible values of μ^2 are dense in the unit interval. Thus, $\frac{c_1^2}{\chi} = \frac{8}{1-\sigma/c_1^2}$ ranges over values dense in $[4, 8]$.

It remains to check the divisibilities of the canonical classes of the surfaces with fixed invariants that we have constructed. The divisibility is $m_0 + (d-1)\sum_j m_j - 3$, because the pullback of the hyperplane class to the iterated covering is primitive, compare [31] Proposition 10 and corollary. Thus the divisibility is odd if and only if m_0 is even. This imposes no restriction on our construction. We obtain even divisibilities if and only if m_0 is odd. Since m_0 is restricted to multiples of the denominators of the μ_j, this requires that these denominators be odd. This restriction still leaves us with a set of attainable μ^2 which is dense in the unit interval.

Now for each $d \in D$ we obtain a unique total covering degree $2^s d^{16}$. This implies that the surfaces obtained for a fixed c_1^2 and different d have different divisibilities $d^8\sqrt{c_1^2/2^s}$ for their canonical classes. □

Corollary 1. *For every $k > 0$ there are k-tuples of simply connected spin and non-spin complex algebraic surfaces with ample canonical bundles which are homeomorphic, but are pairwise non-diffeomorphic. For every k, the ratios c_1^2/χ of such k-tuples are dense in the interval* $[4, 8]$.

Proof. The surfaces in Theorem 3 are all simply connected and have ample canonical bundles. They are spin or non-spin according to whether the divisibility of the canonical class is even or odd. Once the parity of the divisibility and the values of the Chern numbers are fixed, all these surfaces are homeomorphic by Freedman's result [11]. However, surfaces with different divisibilities cannot be diffeomorphic, because Seiberg–Witten theory shows that the canonical class of a minimal surface of general type is diffeomorphism-invariant (up to sign), compare [43], page 789. □

3. GEOGRAPHY OF ACD SYMPLECTIC MANIFOLDS

In this section we study the geography of simply connected minimal symplectic four–manifolds which are almost completely decomposable. Recall that a four–manifold X is called almost completely decomposable or ACD if $X\#\mathbb{C}P^2$ is diffeomorphic to $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ for some p and q. Mandelbaum [29] conjectured that every simply connected complex algebraic surface is ACD, and it is very natural to extend this to simply connected symplectic four–manifolds. Mandelbaum and Moishezon proved that certain algebraic surfaces, including simply connected elliptic surfaces, complete intersections and double planes are indeed ACD, compare [28, 29, 30] and the references cited there.

The geography of minimal symplectic four–manifolds has been investigated by many authors in recent years, for example by Fintushel–Stern [8], Gompf [13], Park [35, 36] and Stipsicz [40]. Here we reprove and improve their results, although we only use manifolds with the ACD property. That the ACD condition does not constrain the geography can be taken as evidence that all simply connected symplectic manifolds may be ACD. The restriction to minimal symplectic manifolds is natural, and implies that all our manifolds will be irreducible, see [18], which gives the ACD property added interest.

We use the coordinates $c_1^2 = 2e + 3\sigma$ and $\chi = \frac{1}{4}(e + \sigma)$ to state our geography results. By the work of Taubes [41, 18], minimal symplectic four–manifolds satisfy $c_1^2 \geqslant 0$. It is clear that in the simply connected case one must have $\chi > 0$. Thus, we try to cover lattice points in the first quadrant of the (χ, c_1^2)–plane with simply connected minimal symplectic manifolds which dissolve after connected sum with only one copy of $\mathbb{C}P^2$. All our manifolds are, or can be made, non-spin, and we will not repeat this. We shall use symplectic summation along submanifolds, as pioneered by Gompf [13], with only a handful of building blocks. Most of these summations will be along tori of zero selfintersection, in which case the Chern invariants are additive. For the resulting manifolds we have to check minimality and the ACD property. Minimality is always true if we sum minimal

symplectic manifolds with $\chi > 1$. Proofs of this have been given by Li–Stipsicz [27] and by Park [35], with the latter attributing the result to Lorek. To prove almost complete decomposability we shall use the following "irrational connected sum lemma" of Mandelbaum, see [28, 29, 12].

Proposition 1. *Let M and N be simply connected oriented 4-manifolds containing the same embedded surface F of genus $g \geqslant 1$ with zero selfintersection. Assume that F has simply connected complement in M. Denote by P the sum of M and N along F, and assume that P is not spin.*

(1) *Then $P\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ is diffeomorphic to $M\#N\#2g(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$.*
(2) *If (N, F) is obtained from a pair (N', F') by blowing up a point on F', then $P\#\mathbb{C}P^2$ is diffeomorphic to $M\#N'\#2g(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$.*

We now list our building blocks.

Example 1 (Elliptic building blocks). We shall denote by $E(n)$ the relatively minimal elliptic surface with $\chi(E(n)) = n$ over S^2 without multiple fibers. This is the fiber sum of n copies of $E(1) = \mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$, and is therefore ACD by the second part of Proposition 1. It is minimal as soon as $n > 1$.

We can use logarithmic transformations on $E(n)$ to produce infinitely many distinct smooth structures, all of which support symplectic structures such that the fibers are symplectic submanifolds. Using such logarithmic transformations we can also change the homeomorphism type of $E(2n)$, which is spin, to a non-spin elliptic surface. All these elliptic surfaces with multiple fibers are ACD, see [28, 30].

Gompf [13] has shown that the $K3$ surface $E(2)$ contains two disjoint nuclei corresponding to different elliptic fibrations in such a way that there is a symplectic form for which the tori in the two nuclei are simultaneously symplectic. This is useful because each of the nuclei contains a 2-sphere intersecting the torus fiber once, so that one can perform logarithmic transformations or symplectic summations independently inside the two nuclei, without introducing a nontrivial fundamental group.

Example 2 (A small building block). We shall denote by S the simply connected symplectic manifold $S_{1,1}$ constructed by Gompf in [13], Example 5.4. It has $c_1^2(S) = 1$, and $\chi(S) = 2$, and contains a symplectically embedded torus T of zero selfintersection and a symplectically embedded genus 2 surface F disjoint from T, such that $S \setminus (T \cup F)$ is simply connected. That S is irreducible was proved by Stipsicz [39]. *A fortiori* it is minimal.

Example 3 (Building blocks with positive signature). We use the following construction of Li–Stipsicz [27], compare also [40].

For every positive integer n there is a symplectic manifold X_n which is a Lefschetz fibration over the surface Σ_{n+2} of genus $n + 2$ which admits a

section of selfintersection $-n-1$. It has Chern invariants

$$\chi(X_n) = 25n^2 + 30n + 1 \ ,$$
$$c_1^2(X_n) = 225n^2 + 180n \ .$$

Furthermore, the fibration and the section induce inverses of each other on the fundamental groups. Thus one can kill the fundamental group of X_n by symplectic summation along the section. We will use a blowup of the $K3$ surface as follows. First we construct a smooth symplectic submanifold inside a nucleus by smoothing the union of $n+2$ copies of a regular fiber and one copy of a section. This gives a surface of genus $n+2$ and selfintersection $2n+2$. Blowing up $n+1$ points on this surface, its selfintersection drops to $n+1$, so that it can be symplectically summed to the section of X_n. Note that the surface has simply connected complement inside the blown-up nucleus of $K3$. In this way we obtain a simply connected minimal symplectic 4–manifold Y_n. Using the above formulae for the Chern invariants of X_n we obtain:

$$\chi(Y_n) = 25n^2 + 31n + 4 \ ,$$
$$c_1^2(Y_n) = 225n^2 + 187n + 7 \ .$$

These manifolds are not spin and still contain a nucleus of a $K3$ surface with simply connected complement. Note that for every $\epsilon > 0$ there is an n such that $c_1^2(Y_n)/\chi(Y_n) > 9-\epsilon$.

As a warmup for our main geography result we first show how to fill up a certain region, which includes that below the Noether line. The importance of this is that the width of this region in the y direction goes to infinity with x.

For a constant c let R_c denote the set of lattice points (x, y) in the plane satisfying $x > 0$, $y \geqslant 0$, and

$$y \leqslant 3x - 51 \ , \tag{9}$$
$$y \leqslant 6x - c \ . \tag{10}$$

Proposition 2. *There exists a constant c such that all lattice points in R_c are realized as the Chern invariants (χ, c_1^2) of infinitely many homeomorphic pairwise nondiffeomorphic simply connected minimal symplectic manifolds, all of which are almost completely decomposable.*

Proof. All our examples will be non-spin and will have the same Chern invariants. Thus they are homeomorphic by Freedman's classification [11].

There is nothing to prove for $y = 0$, as minimal elliptic surfaces and their logarithmic transformations give the required examples. Thus we may assume $y > 0$. Given a positive integer y, we can write it uniquely as $y = 9k + r - 8$, with $0 \leqslant r \leqslant 8$ and $k > 0$. Then consider the manifold

$X(k, r, n)$ obtained as the symplectic sum of k copies of building block S summed along the genus 2 surface F, and of r further copies of S and one copy of $E(n)$ summed to the result along the torus T. This is again simply connected. The Chern invariants are $c_1^2(X(k, r, n)) = 9k + r - 8$ and $\chi(X(k, r, n)) = 3k + 2r + n - 1$. If we take $n \geqslant 2$, then the building blocks are minimal, and so are the $X(k, r, n)$. Moreover, the $X(k, r, n)$ fill out the claimed region (for any c).

Consider now the connected sum $X(k, r, n)\#\mathbb{C}P^2$. Applying the second part of Proposition 1 to the seam inside the elliptic piece $E(n) = E(1) \cup_{T^2} E(n-1)$ we can split off a copy of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Then using this to apply the first part of Proposition 1 to the remaining seams, and breaking up the elliptic pieces, we see that

$$
\begin{aligned}
&X(k, r, n)\#\mathbb{C}P^2 \\
&\qquad\cong (k + r)S\#(3k + r + 2n - 2)\mathbb{C}P^2\#(3k + r + 10n - 3)\overline{\mathbb{C}P^2} \\
&\qquad\cong (k + r)S\#(3k + r + 2n - 2)(S^2 \times S^2)\#(8n - 1)\overline{\mathbb{C}P^2}\ .
\end{aligned}
$$

By the result of Wall [42] there is a k_0 such that $S\#k_0(S^2 \times S^2)$ is completely decomposable. Therefore, $X(k, r, n)\#\mathbb{C}P^2$ dissolves as soon as $3k + r + 2n - 2 \geqslant k_0$, which follows from (10) with $c = 3k_0 + 72$.

It remains to show that there are infinitely many symplectic manifolds homeomorphic but non-diffeomorphic to $X(k, r, n)$, all of which are ACD. For this we replace the elliptic surface $E(n)$ without multiple fibers by one with multiple fibers obtained by logarithmic transformation. In this case the general fiber becomes divisible in homology, in particular its complement is no longer simply connected. Here this is irrelevant because the torus in S has simply connected complement, so that the symplectic sum does give a simply connected manifold and Proposition 1 can be applied.

The logarithmic transformations on $E(n)$ produce infinitely many distinct smooth structures on the topological manifold underlying $E(n)$, which are detected by Seiberg–Witten invariants, cf. [43]. This difference in the Seiberg–Witten invariants survives the symplectic sum operation along a fiber, because of the gluing formulas due to Morgan–Mrówka–Szabó [32] and Morgan–Szabó–Taubes [33]. Thus, we can produce infinitely many minimal symplectic manifolds homeomorphic but non-diffeomorphic to $X(k, r, n)$. All these are ACD by the same argument as for $X(k, r, n)$ (and the fact that the elliptic building blocks are ACD even when they contain multiple fibers). $\square$

Theorem 4. *For every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that every lattice point (x, y) in the first quadrant satisfying*

$$y \leqslant (9 - \epsilon)x - c(\epsilon) \tag{11}$$

is realized by the Chern invariants (χ, c_1^2) *of infinitely many pairwise non-diffeomorphic simply connected minimal symplectic manifolds, all of which are almost completely decomposable.*

Proof. Given $\epsilon > 0$, we first choose an i such that $c_1^2(Y_i)/\chi(Y_i) > 9 - \epsilon$ for the building block Y_i in Example 3. Denote this fixed Y_i by Y.

Let $Y(l, k, r, n)$ be the symplectic manifold obtained by symplectically summing l copies of Y along the torus T in the $K3$ nucleus inside Y, and then summing the result to the manifold $X(k, r, n)$ from the proof of Proposition 2 using the same torus in Y and the torus in $X(k, r, n)$ coming from the elliptic piece. If we choose $c(\epsilon)$ large enough, then all the lattice points satisfying (11) are covered by the translates of R_c which we obtain in this way. In all these summations the complement of the surface along which the summation is performed is simply connected in at least one of the summands, so that the resulting manifolds are simply connected. They are all minimal, as we may assume $n > 1$.

Consider now the connected sum $Y(l, k, r, n)\#\mathbb{C}P^2$. Applying the second part of Proposition 1 to the seam inside the elliptic piece $E(n) = E(1) \cup_{T^2} E(n-1)$ we can split off a copy of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Then using this to apply the first part of Proposition 1 to the remaining seams, and breaking up the elliptic pieces, we see that

$$
\begin{aligned}
&Y(l, k, r, n)\#\mathbb{C}P^2 \\
&\cong lY\#(k + r)S\#(l + 3k + r + 2n - 2)\mathbb{C}P^2\#(l + 3k + r + 10n - 3)\overline{\mathbb{C}P^2} \\
&\cong lY\#(k + r)S\#(l + 3k + r + 2n - 2)(S^2 \times S^2)\#(8n - 1)\overline{\mathbb{C}P^2}\,.
\end{aligned}
$$

By choosing $c(\epsilon)$ large enough, we can ensure that $l + 3k + r + 2n - 2$ is always larger than the "resolving number" of Y and of S, cf. [29]. This means that the result of Wall [42] can be applied to show that the above connected sum is completely decomposable. Thus, if the Chern invariants (x, y) of $Y(l, k, r, n)$ satisfy $y \leqslant (9 - \epsilon)x - c(\epsilon)$ with large enough $c(\epsilon)$, we conclude that $Y(l, k, r, n)$ is ACD.

It remains to show that there are infinitely many symplectic manifolds homeomorphic but non-diffeomorphic to $Y(l, k, r, n)$, all of which are ACD. For this we can replace the elliptic surface $E(n)$ without multiple fibers by one with multiple fibers obtained by logarithmic transformation as in the proof of Proposition 2. As all elliptic surfaces are ACD, and the logarithmic transformations can be assumed to have been made inside $E(n - 1)$ in a splitting $E(n) = E(n - 1) \cup_{T^2} E(1)$, all the resulting manifolds will be ACD by the same argument as above. $\square$

4. The main theorems

The following theorem is very close to various results proved by Mandelbaum and Moishezon, and will be proved using their technique, but it does not appear explicitly in their papers [28, 29, 30]. The case of complete intersections does appear there, and it is pointed out that they are branched covers, but the latter are not treated in complete generality.

Theorem 5. *Iterated branched covers of the projective plane are almost completely decomposable.*

Proof. The proof is by induction on the number of iterations. To begin, note that the cyclic cover of degree d of the complex projective plane branched in a smooth curve of degree $d \cdot m$ is ACD by applying Theorem 2.9 of [29] to the Veronese embedding of $\mathbb{C}P^2$ given by the monomials of degree m.

It remains to show that if $f\colon X \to \mathbb{C}P^2$ is an iterated branched cover and Y is the cyclic branched cover of X branched over $f^{-1}(C_d)$, where C_d is a general plane curve of degree $d \cdot m$, then Y is ACD if X is.

Keeping in mind that $f^{-1}(C_d)$ is ample, one can find closely related statements in [29], yet we cannot rely on them directly. In Theorem 2.9 of [29] the branch locus is assumed to be very ample, whereas Theorem 2.14 refers to the homology class of the branching locus and is not applicable to a given representative. Nevertheless we follow the line of argument of Mandelbaum and Moishezon, see in particular Theorem 4.2 in [28] or Theorems 4.1 and 4.2 in [30].

For each $k \geqslant 0$ choose a general section s_k of the degree km line bundle on the plane and let C_k be its vanishing locus. Then $f^{-1}(C_k)$ is a smooth curve in X given by the equation $f^*s_k = 0$. Consider the line bundle $L = f^*\mathcal{O}(m)$ and its compactification $W = \mathbb{P}(L \oplus \mathcal{O}_X)$. If $p\colon W \to X$ is the bundle projection and $W_\infty = W \smallsetminus L$ is the section at infinity then the line bundle $E = p^*(L)(W_\infty)$ admits a tautological section y without zeroes at infinity. A k-sheeted cyclic covering Y_k of X branched over $f^{-1}(C_k)$ is described in W as the vanishing locus of the section $t_k = y^k - (fp)^*s_k$ of E^k. Consider the pencil in $|E^d|$ generated by t_d and $t_1 \cdot t_{d-1}$. The general member of the pencil is smooth and diffeomorphic to Y_d. The special member given by the vanishing of $t_1 t_{d-1}$ is the union of two smooth surfaces $Y_1 \cong X$ and Y_{d-1} intersecting in a curve which is given on Y_1 by the equation $y^{d-1} = (fp)^*s_{d-1}$. The obvious isotopy from Y_1 to X (embedded into W as the zero section) transforms this curve into the curve on X given by $0 = f^*s_{d-1}$, i. e. $f^{-1}(C_{d-1})$. If this is a sphere then its preimage in Y_{d-1} is a sphere with positive self-intersection. In this case Y_{d-1} is rational hence completely decomposable.

On the other hand, if the curve $f^{-1}(C_{d-1})$ has genus at least 1 we consider the fibration of the blowup of W along $(Y_1 \cup Y_{d-1}) \cap Y_d$ over $\mathbb{C}P^1$ given by our pencil. By [30] Corollary 2.7, Y_d is the sum of a blowup of $Y_1 \cong X$ and of Y_{d-1} along $f^{-1}(C_{d-1})$. By [30] Theorem 2.8 (2) it follows that $Y_d \# \mathbb{C}P^2$ is diffeomorphic to the connected sum of X and Y_{d-1} together with $k > 0$ copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. Since X is almost completely decomposable the result follows by induction. □

Remark 1. The same argument applies to the iterated branched covers of a quadric considered by Moishezon [31].

Next we exhibit manifolds covering a large geographical area satisfying the Hitchin-Thorpe inequality (1), but which have infinitely many smooth structures not supporting Einstein metrics.

Theorem 6. *For every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that every lattice point (x, y) with $y \geqslant 0$ satisfying*

$$y \leqslant (6 - \epsilon)x - c(\epsilon)$$

is realized by the Chern invariants (χ, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected almost completely decomposable symplectic manifolds which do not admit Einstein metrics.

Proof. We consider the manifolds $Y(l, k, r, n)$ in the proof of Theorem 4 above. They are all symplectic, and so have non-trivial Seiberg-Witten invariants. Therefore [24, 19], if such a manifold is blown up sufficiently often, the blowup cannot admit any Einstein metric. According to Theorem 3.3 of LeBrun [26], $\frac{1}{3}c_1^2(Y(l, k, r, n))$ many blowups suffice. Thus, these manifolds cover the claimed area. The infinitely many distinct smooth structures on each remain distinct under blowing up, see for example [9, 22]. Clearly the ACD property is preserved by the blowups. □

We can now combine the results proved so far in order to prove the following more detailed version of Theorem 2.

Theorem 7. *For every natural number k there are simply connected topological 4-manifolds M_k which have at least k distinct smooth structures Z_k^i supporting Einstein metrics, and also have infinitely many distinct smooth structures X_k^j not supporting Einstein metrics.*

The Z_k^i and X_k^j can be chosen symplectic and almost completely decomposable. For every fixed k, the ratios c_1^2/χ of the Chern invariants of such examples are dense in the interval $[4, 6]$.

Proof. We consider certain simply connected symplectic manifolds which are non-spin and have the same Chern invariants. Thus they are homeomorphic by Freedman's classification [11].

The Z_k^i are the iterated branched covers of the projective plane constructed in Theorem 3. By Corollary 1, the ratios of the Chern invariants of such examples are dense in the interval $[4, 8]$. By Theorem 5, these manifolds are ACD. As they are Kähler with ample canonical bundle, the solution of the Calabi conjecture due to Aubin [1] and Yau [44] shows that they carry Einstein metrics.

Bringing down the upper bound for the slope to 6 allows us to use manifolds from Theorem 6 having appropriate Chern invariants for the X_k^j. These are ACD by construction and do not carry Einstein metrics.

We already noted in Corollary 1 that the Z_k^i are pairwise non-diffeomorphic by Seiberg–Witten theory [43]. The X_k^j are obtained by blowing up distinct smooth structures distinguished by their Seiberg-Witten invariants, and so they are also distinct because of the blowup formula [9, 22]. Clearly no Z_k^i can be diffeomorphic to a X_k^j, as the former admit Einstein metrics and the latter do not. (Also, the former are irreducible [18], and the latter are not.) □

Remark 2. The manifolds M_k have another infinite sequence of smooth structures, which are very likely distinct from the Z_k^i and the X_k^j. Fintushel–Stern [8] showed that one can perform cusp surgery on a torus in any iterated branched cover of the plane to construct infinitely many distinct smooth structures with non-trivial Donaldson invariants. It seems that these are irreducible, and therefore distinct from the X_k^j. On the other hand they are not complex, and therefore distinct from the Z_k^i. Whether they are ACD or admit Einstein metrics is not known.

Theorems 4 and 6 also lead to the following more general existence result for smooth structures not supporting Einstein metrics.

Theorem 8. *For every $\epsilon > 0$ there is a constant $c(\epsilon) > 0$ such that the connected sum $p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$ has infinitely many smooth structures not admitting Einstein metrics for every large enough $p \not\equiv 0 \pmod 8$ and $q \geqslant (2+\epsilon)p + c(\epsilon)$.*

Proof. For odd p, this was already proved in Theorem 6.

For even p, we are in a situation where the numerical Seiberg–Witten invariants must vanish. Therefore, to obtain an obstruction to the existence of Einstein metrics one considers the refined Seiberg–Witten invariants of Bauer–Furuta [3] in the context of stable homotopy theory. Using this approach, Ishida–LeBrun [16] showed that a connected sum $X_1\#X_2\#k\overline{\mathbb{C}P^2}$, where the X_i are simply connected symplectic four-manifolds with $b_2^+ \equiv 3 \pmod 4$, does not admit Einstein metrics if $k \geqslant \frac{1}{3}(c_1^2(X_1) + c_1^2(X_2)) - 4$. Applying this to the case where X_1 are the manifolds from Theorem 4 with $b_2^+ \equiv 3 \pmod 4$ and X_2 is the $K3$ surface, proves the claim of the

Theorem for $p \equiv 2 \pmod 4$. We just have to see that the connected sum with $K3$ does not collapse the infinitely many smooth structures on $X_1 \# k\overline{\mathbb{C}P^2}$. These smooth structures were constructed by logarithmic transformation on an elliptic building block in X_1. As we increase the multiplicity of the logarithmic transformation, we find that there are more and more Seiberg–Witten basic classes whose numerical Seiberg–Witten invariants are ± 1, see Fintushel–Stern [10], Theorem 8.7. By the result of Bauer [2], these basic classes give rise to monopole classes in the sense of [23] on $X_1 \# X_2 \# k\overline{\mathbb{C}P^2}$. As the expected dimension of the Seiberg–Witten moduli space is positive for all these monopole classes, each smooth structure has at most finitely many such classes. This shows that we have an infinite set of smooth structures[1].

It remains to deal with the case $p \equiv 0 \pmod 4$. The above argument generalizes to the case of connected sums of 4 symplectic manifolds X_i with $b_2^+ \equiv 3 \pmod 4$ as long as the resulting manifold $X_1 \# \ldots \# X_4 \# k\overline{\mathbb{C}P^2}$ has b_2^+ not divisible by 8 and $k \geqslant \frac{1}{3}(c_1^2(X_1) + \ldots + c_1^2(X_4)) - 12$. This was noted by Ishida–LeBrun in [17], using [2]. We apply it here taking for X_1 the manifolds from Theorem 4 with $b_2^+ \equiv 3 \pmod 8$, and taking the $K3$ surface for X_2, X_3 and X_4. This proves the claim of the Theorem for $p \equiv 4 \pmod 8$. □

Remark 3. Theorem 8 should be compared to Theorems 11 and 12 of Ishida–LeBrun [16], which give much weaker statements in the same direction. Namely, if p is odd they assumed $p \equiv 1 \pmod 4$ and $q > \frac{7}{3}p + 12$, which is more restrictive than $q \geqslant (2+\epsilon)p + c(\epsilon)$ for almost all p whenever $\epsilon < \frac{1}{3}$. The unknown constant $c(\epsilon)$ only appears in our statement because we constructed all manifolds to be ACD. If we gave up this constraint, we could make the constant explicit. However, our method of proof, and the smooth structures under consideration, are very different. In our proof, for odd p the smooth structures in question support symplectic forms, and, therefore [18], can not decompose as smooth connected sums except for the splitting off of copies of $\overline{\mathbb{C}P^2}$. The non-existence of Einstein metrics is detected by the numerical Seiberg–Witten invariants. The smooth structures discussed by Ishida–LeBrun [16] are smooth connected sums where each summand has positive b_2^+, and so in particular they cannot support symplectic forms. As the numerical Seiberg–Witten invariants of these smooth structures vanish, the non-existence of Einstein metrics can only be detected using the stable homotopy refinement [3] of the Seiberg–Witten invariants.

For even p, Ishida–LeBrun [16] assumed $p \equiv 2 \pmod 4$ and $q > \frac{7}{3}p + 16$. In this case our proof is similar to theirs—in fact we use the main result of their paper. Our improvement is due to the fact that our Theorem 4

[1]See [21] for more details and elaborations on this argument.

above gives us more symplectic manifolds we can use as connected summands, whereas Ishida and LeBrun used only certain manifolds constructed by Gompf [13] with smaller slope of their Chern invariants.

5. FURTHER EXAMPLES

Since the proof of Theorem 1 in [19], other examples of manifolds with a smooth structure supporting an Einstein metric and one or several without an Einstein metric have appeared, and some attempts have been made to give examples with smallish homology, compare [26, 16]. Here are the ultimate examples, whose second Betti number is a fraction of that of the smallest previously known examples.

Proposition 3. *The manifolds* $3\mathbb{C}P^2\#17\overline{\mathbb{C}P^2}$ *and* $3\mathbb{C}P^2\#18\overline{\mathbb{C}P^2}$ *each have a smooth structure supporting an Einstein metric, and infinitely many smooth structures not supporting Einstein metrics.*

Proof. Take a double cover of $\mathbb{C}P^2$ branched in the union of two smooth cubics in general position. This gives a singular $K3$ surface with 9 nodes. Now take a further double cover of this singular $K3$ surface branched in the nodes and in the preimage of a line. This gives a simply connected smooth algebraic surface S with ample canonical bundle, whose numerical invariants are $c_1^2(S) = 1$, $\chi(S) = 2$, compare [4]. It is homeomorphic to $3\mathbb{C}P^2\#18\overline{\mathbb{C}P^2}$ by Freedman's classification [11]. By the results of Aubin [1] and Yau [44] it admits a Kähler–Einstein metric.

One way to obtain a smooth manifold homeomorphic to S which cannot admit an Einstein metric is to take a simply connected algebraic surface S' with $c_1^2(S') = 2$, $\chi(S') = 2$, and blow it up once. Such S' exist, compare Catanese–Debarre [6] and the discussion below. According to LeBrun [26], the blowup of S' does not admit any Einstein metric. Another possibility is to take a symplectic manifold homeomorphic to $3\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ for some $n \leqslant 16$, and blow it up until it becomes homeomorphic to S. Such manifolds have been constructed by Gompf [13] and D. B. Park [34]. There are in fact infinite sets of smooth structures on them supporting symplectic forms, compare [36]. These remain distinct under blowing up, and the blowups have no Einstein metrics by the result of [24, 19], say. This proves the claim for $3\mathbb{C}P^2\#18\overline{\mathbb{C}P^2}$.

To obtain an algebraic surface homeomorphic to $3\mathbb{C}P^2\#17\overline{\mathbb{C}P^2}$ which has ample canonical bundle one can proceed as follows. Take a double cover of $\mathbb{C}P^1 \times \mathbb{C}P^1$ branched in the union of two smooth curves of bidegrees $(3,1)$ and $(1,3)$ respectively. Then take a further double cover branched in the nodes of the first covering and the preimage of a smooth curve of bidegree $(1,1)$ in general position with respect to the other two curves. The

resulting smooth surface S' has all the desired properties, compare [6]. If we start with one of the symplectic manifolds with $c_1^2 \geqslant 6$ constructed by D. B. Park [34], then there are infinitely many smooth structures on it which remain distinct under blowing up, and the blowups homeomorphic to S' do not admit any Einstein metrics by [24, 19]. □

Catanese [4] proved that all the algebraic surfaces S homeomorphic to $3\mathbb{C}P^2 \# 18\overline{\mathbb{C}P^2}$ are diffeomorphic to each other. In [6] it is conjectured that the same is true for surfaces homeomorphic to $3\mathbb{C}P^2 \# 17\overline{\mathbb{C}P^2}$. Thus, one has to take larger examples to obtain multiple smooth structures with Einstein metrics[2]. While our proof of Theorem 2 can be made effective, in practice the manifolds one obtains will have huge homology. Nevertheless, concrete examples can be given quite easily.

Example 4. For any integer $k \geqslant 0$ let Z_2^1 be a smooth hypersurface of bidegree $(5+k, 6)$ in $\mathbb{C}P^1 \times \mathbb{C}P^2$, and let Z_2^2 be a smooth complete intersection of two hypersurfaces of bidegrees $(2,1)$ and $(1+k, 6)$ in $\mathbb{C}P^1 \times \mathbb{C}P^3$. Both have

$$c_1^2 = 9(17+5k) \ ,$$
$$\chi = 41 + 10k \ .$$

The divisibility of the canonical class is $gcd\{k+3, 3\} = gcd\{k, 3\}$ for Z_2^1 and $gcd\{k+1, 3\}$ for Z_2^2. Thus they are both non-spin and are homeomorphic for each k. If $k \equiv 1 \pmod 3$, then they both have divisibility $= 1$, otherwise they have different divisibilities, and can therefore not be diffeomorphic.

These surfaces have ample canonical bundles, and therefore [1, 44] support Kähler–Einstein metrics. Their ratio c_1^2/χ is < 4.5, and thus they are well within the range where Theorem 6 shows that there are infinitely many homeomorphic smooth manifolds without Einstein metrics.

Finally, as the manifolds discussed so far are all non-spin, it is worthwhile to point out the following:

Theorem 9. *For every k there is a topological spin four–manifold M_k admitting at least k distinct smooth structures which support Einstein metrics, and which are almost completely decomposable. The ratios $|\sigma|/e$ of such manifolds are dense in the interval $[\frac{1}{3}, \frac{1}{2}]$.*

Proof. This follows from the spin case of Corollary 1 together with the existence result for Kähler–Einstein metrics due to Aubin [1] and Yau [44]. The ACD property was proved in Theorem 5. □

[2] It is known that $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$ has two distinct smooth structures supporting Einstein metrics, see [7, 19], but it is not known whether this manifold has a smooth structure without an Einstein metric.

Remark 4. Such examples are also provided by Catanese's k-tuples of homeomorphic spin surfaces with different divisibilities of their canonical classes constructed as bidouble covers of a quadric [5]. For almost all choices of the parameters, those surfaces have ample canonical bundles. However, the spread of their numerical invariants is probably more restricted than in our examples based on Salvetti's construction [37].

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